Distance domination in partitioned graphs

Fu, Chin-Mei Kau; Vestergaard, Preben D.

Publication date:
2007

Document Version
Publisher's PDF, also known as Version of record

AALBORG UNIVERSITY

Distance domination in partitioned graphs

by

Chin-Mei Kau Fu and Preben Dahl Vestergaard

R-2007-05 February 2007

DEPARTMENT OF MATHEMATICAL SCIENCES
AALBORG UNIVERSITY
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 96 35 80 80 ▪ Telefax: +45 98 15 81 29
URL: www.math.auc.dk/research/reports/reports.htm

ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

Distance domination in partitioned graphs

Chin-Mei Kau Fu
Department of Mathematics, Tamkang University
Tamsui Taipei Shien 25137, Taiwan.
e-mail: cmfu@mail.tku.edu.tw

Preben Dahl Vestergaard
Department of Mathematical Sciences, Aalborg University
Fredrik Bajers Vej 7G, DK-9220 Aalborg Ø, Denmark
email: pdv@math.auc.dk

To appear in **Congressus Numerantium**

Keywords: Domination, partition, tree.
AMS Mathematics Subject Classification: 05C75

Abstract For a graph G with its vertex set partitioned into, say two sets $V(G) = V_1 \cup V_2$, bounds for $\gamma(G) + \gamma_G(V_1) + \gamma_G(V_2)$ have earlier been considered. This is generalized. We define a vertex set to distance d dominate all vertices at distance at most d from it. For partitioned graphs and any $d \geq 2$ we generalize theorems about ordinary distance one domination to distance d domination. Further, we give bounds for distance 2 domination of a graph partitioned into three sets and state a conjecture.

Definitions For $d \geq 1$ the vertex x in a graph is said to *distance d dominate* itself and all vertices at distance at most d away from x. A set D of vertices *distance d dominate* D and all vertices having distance at most d to D. The *distance d domination number* $\gamma_{\leq d}(G)$ of the graph G is the cardinality of a smallest set D which distance d dominates all vertices in G. For $d = 1$ we get the usual domination number $\gamma_{\leq 1}(G) = \gamma(G) = |D|$.

Let $k \geq 2$ be an integer and $V_1, V_2, \ldots, V_k$ a partition of $V(G)$. For $i, 1 \leq i \leq k$, we shall by $\gamma_{\leq d}(G, V_i)$ denote the order of a smallest set of vertices in G which distance d dominates V_i. I.e. there exists $D_i \subseteq V(G)$ such that every vertex of V_i either belongs to D_i or in G has distance at most d to a vertex in D_i, and $\gamma_{\leq d}(G, V_i) = |D_i|$ for a smallest such D_i. Let $f_{\leq d}(k, G)$ denote the maximum taken over all partitions $V_1, \ldots, V_k$ of $V(G)$ of the sum $\gamma_{\leq d}(G) + \sum_{i=1}^{k} \gamma_{\leq d}(G, V_i)$.

For $d = 1$ we write $\gamma(G, V_i)$ and $f(k, G)$. When no misunderstanding is possible we may write $\gamma_{\leq d}(V_i), f_{\leq d}(G)$ for short. Hartnell and Vestergaard gave upper bounds for $f_{\leq d}(k, G) = \gamma_{\leq d}(G) + \sum_{i=1}^{k} \gamma_{\leq d}(G, V_i)$, when $d = 1$. We shall generalize to $d \geq 1$.

For $d = 1$ and $k = 2$ we can slightly reformulate their result:

Theorem 1. *[2] Let G be a graph with at least 3 vertices in each component and let V_1, V_2 be any partition of $V(G)$. Then*

$$\gamma(G) + \gamma(G, V_1) + \gamma(G, V_2) \leq \frac{5}{4}|V(G)|, \ \textit{i.e.}\ f(2, G) \leq \frac{5}{4}|V(G)|.$$

Equality occurs if and only if each component of G satisfies

(i) *the number of vertices is a multiple of four.*

(ii) *Every vertex has degree one or is adjacent to exactly one degree one vertex.*

(iii) *Every vertex of degree three or more is adjacent to exactly one degree two vertex having a degree one neighbour.*

(iv) *All degree one vertices are in one class V_1, all degree two vertices in the other class V_2 and vertices of degree ≥ 3 can be in either class.*

For $d \geq 2$ we have Theorem 2 below.

Theorem 2. *Let $d \geq 2$ and let G be a graph with at least $d + 2$ vertices in each component. For any partition $V_1,\ V_2$ of $V(G)$ we have*

$$\gamma_{\leq d}(G) + \gamma_{\leq d}(G, V_1) + \gamma_{\leq d}(G, V_2) \leq \frac{6}{2d + 3}|V(G)|$$

and equality holds if and only if

(i) *the order of each component if G is a multiple of $2d + 3$ and*

(ii) *G can be constructed from a set of disjoint paths of lengths $2d + 2$ by arbitrarily adding edges between their central vertices.*

Proof of inequality.

It suffices to prove the inequality of Theorem 2 for trees. We shall use induction on $n = |V(G)|$.

The inequality is true for $n = d + 2$, for consider, in fact, any tree T on $n \geq d + 2$ vertices and with diameter at most $2d$; then $f_{\leq d}(2, T) \leq 3$, as we can place 3 dominators in the central vertex, when the diameter is an even number, and in an end vertex of the central edge when the diameter of T is an odd number. Obviously $3 \leq \frac{6}{2d+3}(d + 2) \leq \frac{6}{2d+3}n$, so the inequality holds for small values of n.

Assume the inequality to be true for trees with fewer than n vertices. If T has diameter $\geq 2d + 3$ there is an edge e in T such that $T - e$ consists of two trees each having at least $d + 2$ vertices and the inequality holds. So we may assume T has diameter $2d + 1$ or $2d + 2$.

Case 1. Diam $(T) = 2d+1$.

Let $P = v_1 v_2 \dots v_{2d+2}$ be a diametrical path in T. If $T = P$, let $D = \{v_{d+1}, v_{2d+2}\}$, let D_1, D_2 both contain v_{d+1} and place v_{d+2} in D_i if $v_{d+2} \in V_i$, $i = 1, 2$.

Then D dominates $V(T)$, D_i dominates V_i for $i = 1, 2$, and $f_{\leq d}(T) \leq 5$. That satisfies the inequality as $d \geq 2$ implies $5 \leq \frac{6}{2d+3}(2d+2)$.

Otherwise, $n \geq 2d+3$ and with $D = D_1 = D_2 = \{v_{d+1}, v_{d+2}\}$ we obtain

$$f_{\leq d}(2, T) \leq 6 \leq \frac{6}{2d+3} n.$$

Case 2. Diam $(T) = 2d+2$.

Let $P = v_1 v_2 \dots v_{2d+3}$ be a diametrical path of T. If $\deg(v_i) \geq 3$ for any $i \neq d+2$ there is in T an edge e such that the two trees of $T - e$ both have $\geq d+2$ vertices and the inequality holds.

So we may assume that on P no other vertex than v_{d+2} has degree more than two. Assume $T - E(P)$ contains a path $v_{d+2} x_1 x_2 \dots x_{d+1}$. If $\deg(x_j) \geq 3$ for any j, $1 \leq j \leq d$, the two trees in $T - v_{d+2} x_1$ both have $\geq d+2$ vertices and the inequality holds. So we may assume that $\deg(x_j) = 2$ for $1 \leq i \leq d$.

Thus T contains α paths, $\alpha \geq 2$, each of length $d+1$ and pendent from the central vertex v_{d+2} and possibly T also has other vertices, they all are within distance d from v_{d+2}.

Case 2A. Assume T consists of α paths of length $d+1$ pendent from v_{d+2}. Then $n = |V(T)| = 1 + \alpha(d+1)$ and we see that $f_{\leq d}(2, T) \leq 2\alpha + 2$ by placing α vertices adjacent to v_{d+2} in D, placing v_{d+2} in both D_1 and D_2 and placing the α vertices at distance $d+1$ from v_{d+2} in D_i when they belong to V_i, $i = 1, 2$. We certainly have $2\alpha + 2 \leq \frac{6}{2d+3}(\alpha d + \alpha + 1)$ as $\alpha \geq 2$.

Case 2B. Assume T consists of α paths of length $d+1$ pendent from v_{d+2} and also of vertices $y_1, y_2, \dots, y_t$, $1 \leq t$, such that for $1 \leq i \leq t$, y_i has distance $\leq d$ from v_{d+2}.

Note that those of $y_1, y_2, \dots, y_t$, $1 \leq t$ which are within distance $d-1$ from v_{d+2} are dominated by the D-dominators already chosen in Case 2A. For the remaining vertices y_i at distance d from v_{d+2} there exists in T a path $v_{d+2} y_1 y_2 \dots y_d$ and we have $n \geq 1 + \alpha(d+1) + d$. Taking the dominators from case 2A together with v_{d+2} added to D we obtain

$$f_{\leq d}(2, T) \leq 2\alpha + 3 \leq \frac{6}{2d+3}(1 + \alpha(d+1) + d) \leq \frac{6}{2d+3} n.$$

This proves the inequality of Theorem 2. Finally, let $f_{\leq d}(2, G) = \frac{6}{2d+3}|V(G)|$. Then deletion of edges from G to obtain a tree and smaller trees in the process of proving the inequality of Theorem 2 must at every stage preserve equality, therefore the final components are paths P_{2d+3} and if additional edges have ends at other vertices than centers of these paths, we get inequality. This proves Theorem 2. ∎

Comment. *The bound of Theorem 2 is best possible, but only slightly better than the crude evaluation* $f_{\leq d}(2,G) \leq 3 \cdot \gamma_{\leq d}(G) \leq 3\frac{1}{d+1}|V(G)|$. *(cf. [4])*

For partition into 3 classes, a best possible inequality is given by Hartnell and Vestergaard [2].

Theorem 3. [Hartnell, Vestergaard 2003] *Let $n \geq 3$ be an integer. Let T be a tree on n vertices such that $T \notin \{P_4, P_7\}$ and let $\{V_1, V_2, V_3\}$ be a partition of $V(T)$. Then*

$$\gamma(T) + \gamma_T(V_1) + \gamma_T(V_2) + \gamma_T(V_3) \leq \frac{7n}{5}.$$

For distance 2 domination of a tree T with its vertex set partitioned into 3 sets we shall prove.

Theorem 4. *Let $n \geq 4$ be an integer. Let T be a tree on n vertices and let $\{V_1, V_2, V_3\}$ be a partition of $V(T)$. Then*

$$\gamma_{\leq 2}(T) + \gamma_{\leq 2}(V_1) + \gamma_{\leq 2}(V_2) + \gamma_{\leq 2}(V_3) \leq n.$$

Proof. It is enough to prove the theorem for trees. By induction on n it is enough to prove the theorem for trees T with diameter ≤ 6, since otherwise, T has an edge e such that both trees in $T - e$ have ≥ 4 vertices. If T has diameter 2 or 4 it suffices to place its central vertex in each of D, D_1, D_2, D_3. Similarly, if T has diameter 3 we can place an end vertex of the central edge in each of the four dominating sets. In these cases we have $f_{\leq 2}(3,T) \leq 4 \leq n$.

If T has diameter 5, let $v_1 \ldots v_6$ be a diametrical path. Place 4 dominators in v_4 and for each vertex x at distance 3 from v_4, $x \in V_i$, place a D_i-dominator in x and a D-dominator in b, the second last vertex on the unique path $xabv_4$ from x to v_4. In all cases we obtain $f_{\leq 2}(3,T) \leq n$.

Assume T has diameter 6. Let $P = v_1v_2 \ldots v_7$ be a diametrical path in T. If $\deg(v_i) \geq 3$ for $i \neq 4$ there is an edge e in T such that the two trees in $T - e$ have at least 4 vertices and by induction the result follows. So we may assume that $\deg(v_2) = \deg(v_3) = \deg(v_5) = \deg(v_6) = 2$. We easily see that a path P_7 on seven vertices has $f_{\leq 2}(3, P_7) = 7$, i.e. P_7 satisfies Theorem 4, so assume $\deg(v_4) \geq 3$.

Let l denote the length of a longest path emanating from v_4 in $T - E(P)$, $l \leq 3$. For $l = 1$ we place 4 dominators in each of v_3, v_5. For $l = 2, 3$ we place 4 dominators in v_4 and each vertex x at distance 3 from v_4 is chosen to class-dominate itself, while we on $xabv_4$, the unique path from x to v_4 choose b for D-domination. That gives $f_{\leq 2}(3,T) \leq n$. This proves Theorem 4. ■

The inequality of Theorem 4 is best possible as shown by the following examples.

$f_{\leq 2}(3,P_7) = 7$, $f_{\leq 2}(3,P_8) = 8$.

A path on 9 vertices with a pendent edge from its central vertex has $f_{\leq 2}(3,T) = 10 = n$.

However, it can be proven that $f_{\leq 2}(3,T_{11}) \leq 10$ for any tree on n vertices and $f_{\leq 2}(3,T_{12}) \leq 11$ for any tree on 12 vertices. For any tree T_{13} on 13 vertices we have $f_{\leq 2}(3,T_{13}) \leq 12$. So possibly there is a stronger inequality for trees with sufficiently many vertices. Some references to domination in partitioned graphs are given below.

References

[1] B.L. Hartnell & P.D. Vestergaard: "Dominating sets with at most k components", Ars Combinatoria 74 (2005), 223-229.

[2] B.L. Hartnell & P.D. Vestergaard: "Partitions and domination in a graph", JCMCC, Journal of Combinatorial Mathematics and Combinatorial Computing 46 (2003), 113-128.

[3] B.L. Hartnell & D.F. Rall: "Improving some bounds for dominating cartesian products", Discussiones Mathematicae Graph Theory, 23 (2003), 261-272.

[4] M.A. Henning: "Distance domination in graphs", chapter 12 pp. 321-349 in T.W. Haynes, S.T. Hedetniemi, P.J. Slater: "Domination in graphs, advanced topics", Marcel Dekker, Inc. New York 1998.

[5] S. Seager: "Partition domination of graphs of minimum degree two". Congressus Numerantium 132, 1998, 85–91.

[6] Zs. Tuza & P.D. Vestergaard: "Domination in partitioned graphs", Discussiones Mathematicae Graph Theory 22 (2002), 199-210.